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AN ODD CHARACTERIZATION OF J_4

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ABSTRACT

Z. Janko recently discovered a finite simple group called J_4. The purpose of this paper is to classify J_4 by the structure of the centralizer of an element of order three.

It seems probable that in the near future the problem of determining all finite simple groups will be reduced to determining the simple groups all of whose 2-local subgroups are 2-constraint. For finite simple groups possessing a non 2-constraint 2-local subgroup we almost have a standard subgroup. This was a key subgroup for the classification of these groups. If G is a known finite simple group all of whose 2-locals are 2-constraint there almost is a p-standard subgroup A for a certain prime p. It seems likely that in general there is such a p-standard subgroup or G is known.

A subgroup A of G is said to be p-standard if $A' = A$ and $A/Z(A)$ simple. Further $C_G(A)$ possesses nontrivial cyclic Sylow p-subgroups. If $P \in \mathrm{Syl}_p(C_G(A))$, then $N_G(\langle x \rangle) \leqq N_G(A)$ for all $1 \neq x \in P$. We finally have some connection between p and A.

Thus it seems to be a good concept to classify finite simple groups by a p-standard subgroup. In working with a standard subgroup A such that $A/Z(A) \cong M_{22}$, the prime in question is three, it is necessary to classify J_4 by the centralizer of an element of order three which is the full covering group of M_{22}. This classification might be of independent interest. The result reads as follows.

THEOREM. *Let G be a finite group of characteristic 2 type and ρ an element of order three in G such that $O^{\{2,3\}'}(C_G(\rho))$ is isomorphic to $\hat{M}_{22}$ the full covering group of M_{22}. If G is not 3-normal then G is isomorphic to J_4.*

A finite group is said to be of characteristic 2 type iff all 2-local subgroups are corefree and 2-constraint. I hope all other notations will be standard.

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First of all we list some properties of M_{22}, $\hat{M}_{22}$ and $\mathrm{Aut}(M_{22})$ which can be found in [2].

LEMMA 1. (i) *There is exactly one conjugacy class of involutions in* M_{22}. *There are involutions in* $\mathrm{Aut}(M_{22})-M_{22}$.

(ii) *Let* T *be a Sylow* 2-*subgroup of* M_{22}. *Then* $T=\Omega_1(T)$. *There are precisely two elementary abelian subgroups* E_1 *and* E_2 *of order* 16 *in* T. *We have* $N(E_1)/E_1\cong A_6$ *while* $N(E_2)/E_2\cong\Sigma_5$. *Let* T_1 *be a Sylow* 2-*subgroup of* $\mathrm{Aut}(M_{22})$ *containing* T. *Then* $C_{T_1}(E_1)=E_1$ *while* $C_{T_1}(E_2)\neq E_2$.

(iii) *All elements of order three in* $\hat{M}_{22}-Z(\hat{M}_{22})$ *are conjugate. Let* τ *be an element of order three in* $\hat{M}_{22}-Z(\hat{M}_{22})$. *Then a Sylow* 2-*subgroup of* $C_{\hat{M}_{22}}(\tau)$ *is of order* 8.

(iv) *Let* S *be a Sylow* 3-*subgroup of* $\hat{M}_{22}$. *Then* $|S|=27$ *and* $|Z(S)|=3$.

(v) *Let* V *be a Sylow* 5-*subgroup of* M_{22}. *Then* $N_{M_{22}}(V)$ *is a Frobenius-group of order* 20.

(vi) *A Sylow* 11-*normalizer in* M_{22} *is a Frobenius group of order* 55.

LEMMA 2. (i) $C_G(\rho)$ *contains a Sylow* 3-*subgroup of* G.

(ii) *All elements of order three are conjugate in* G.

(iii) *We have* $N_G(\langle\rho\rangle)/Z^*(C_G(\rho))\cong\mathrm{Aut}(M_{22})$.

PROOF. (i) follows from Lemma 1 (iv). Let S be a Sylow 3-subgroup of $C_G(\rho)$. Since G is not 3-normal there is an element $\tau\in S-Z(S)$, $\tau\sim\rho$ in G. By Lemma 1 (iii) we have that all elements of order three in S are conjugate to ρ or ρ^{-1}. As $\tau\sim\tau^{-1}$ in $C_G(\rho)$ we have that all elements of order three are conjugate to ρ. This yields (ii). As $\rho\sim\rho^{-1}$ we have $|N_G(\langle\rho\rangle): C_G(\rho)|=2$. So we have (iii).

LEMMA 3. *Let* z *be an involution in* $Z(C_G(\rho))$. *Then* $Q=O_2(C_G(z))$ *is extraspecial of width* 6. *Further* $C_G(\rho)=O^{\{2,3\}'}(C_G(\rho))$.

PROOF. Set $\tilde{Q}=Q/\langle z\rangle$. As $C_G(\rho)\subseteq C_G(z)$ we get $C_{\tilde{Q}}(\rho)=1$. Let S be a Sylow 3-subgroup of $C_G(\rho)$ and $W=\langle\rho,\tau\rangle$ an elementary abelian subgroup of S of order 9. Let $1\neq\tilde{Q}_1$ be a $N_G(\langle\rho\rangle)$ — invariant subgroup of $\tilde{Q}$. Then W acts on $\tilde{Q}_1$. The fixpoint-formula yields

$$|\tilde{Q}_1|\,|C_{\tilde{Q}_1}(W)|^3=|C_{\tilde{Q}_1}(\langle\tau\rangle)|\,|C_{\tilde{Q}_1}(\langle\rho\rangle)|\,|C_{\tilde{Q}_1}(\langle\rho\tau\rangle)|\,|C_{\tilde{Q}_1}(\langle\rho\tau^{-1}\rangle)|.$$

From Lemma 1 (iii) it follows that $\tau\sim\rho\tau\sim\rho\tau^{-1}$ in $N_G(\langle\rho\rangle)$. Hence $|\tilde{Q}_1|=|C_{\tilde{Q}_1}(\langle\tau\rangle)|^3$. From Lemma 1 (iii) and Lemma 2 it follows that a Sylow 2-subgroup of $C_G(\tau)\cap N_G(\langle\rho\rangle)$ is of order 16. Set $H=\langle Q,N_G(\langle\rho\rangle)\rangle$. Then a Sylow 2-subgroup of $C_H(\tau)$ is of order $8\cdot|C_Q(\tau)|$. As τ is conjugated to ρ in G we get that a Sylow 2-subgroup of $C_G(\tau)$ is of order 2^8. This yields $|C_Q(\tau)| \leqq 2^5$. Now ρ acts on $C_{\tilde{Q}_1}(\tau)$. Further $C_{C_{\tilde{Q}_1}(\tau)}(\rho) = 1$. Hence $C_{\tilde{Q}_1}(\tau)$ is of order 4 or 16. Therefore $\tilde{Q}_1$ is of order 2^6 or 2^{12}. As $11 \nmid |GL(6,2)|$, $|\tilde{Q}_1| = 2^{12}$. Now we have $|C_{\tilde{Q}_1}(\tau)| = 2^4 = |C_{\tilde{Q}}(\tau)|$. This yields $Q_1 = Q$. Hence $\tilde{Q}$ is an irreducible H/Q-module. Then Q is extraspecial of width 6 or elementary abelian. Assume the latter. Set $H_1 = C_G(\tau)$ and $V = C_Q(\tau)$. Then V is elementary abelian of order 2^5. As $V/Z(H_1)$ is elementary abelian of order 16 Lemma 1 (ii) yields $N_{H_1}(V)/C_{H_1}(V) \cong A_6$ or Σ_5. Now $N_{H_1}(V)$ contains $C_{H_1}(\rho)$. From Lemma 1 (iii) it follows that $C_{H_1}(\rho)$ contains a subgroup $X \times \langle \rho \rangle$ where X is of order 8. Hence $N_{H_1}(V)/C_{H_1}(V)$ contains a subgroup $X_1 \times Y$ where Y is of order 3 and X_1 of order 4. But this contradicts the structure of A_6 and Σ_5. This completes the proof of the lemma.

Lemma 4. *Set $H = \langle Q, N_G(\langle \rho \rangle)\rangle$ and $H_1 = \langle Q, C_G(\rho)\rangle$. Let x be in H_1, $x^2 \in Q$. If $x^h \in H$ for some $h \in C_G(z)$ then $x^h \in H_1$.*

Proof. Set $\bar{H} = H/Q$ and $\tilde{H} = H/\langle z \rangle$. Let ν be in H_1, $o(\nu) = 11$. Then $|C_{\tilde{Q}}(\nu)| = 4$. According to Lemma 1 (vi) choose $\omega \in N_H(\langle \nu \rangle)$, $o(\omega) = 5$. Then $[C_{\tilde{Q}}(\nu), \omega] = 1$. Further $|[\tilde{Q}, \omega]| = 16$ or 256. As $\langle \nu, \omega \rangle = \langle \omega, \omega^\nu \rangle$, we get $|[\tilde{Q}, \omega]| = 2^8$. According to Lemma 1 (v) choose $\bar{y} \in N_{\bar{H}_1}(\langle \bar{\omega} \rangle)$, $o(\bar{y}) = 4$. Then $|C_{[\omega, \tilde{Q}]}(\bar{y}^2)| \geqq 2^4$ and $|C_{C_{\tilde{Q}}(\omega)}(\bar{y}^2)| \geqq 4$. As $[\bar{y}^2, \bar{\rho}] = 1$, ρ acts on $C_{C_{\tilde{Q}}(\omega)}(\bar{y}^2)$. This yields $[C_{\tilde{Q}}(\omega), \bar{y}^2] = 1$ or $|C_{C_{\tilde{Q}}(\omega)}(\bar{y}^2)| = 4$. Assume the latter. Since $C_{C_{\tilde{Q}}(\omega)}(\bar{y})$ is $\bar{\rho}$-invariant, we have $C_{C_{\tilde{Q}}(\omega)}(\bar{y}) = C_{C_{\tilde{Q}}(\omega)}(\bar{y}^2)$. Application of [4, lemma (2.1)] yields a contradiction. So we have $|C_{\tilde{Q}}(\bar{y}^2)| \geqq 2^8$. By lemma 1 (i) we get $|C_{\tilde{Q}}(\bar{x})| \geqq 2^8$. Now assume $f = x^h \in H - H_1$. Then $\bar{\rho}^{\bar{f}} = \bar{\rho}^{-1}$. Hence $|C_{\tilde{Q}}(\bar{f})| = 2^6$. But this is impossible.

Lemma 5. *$H = \langle Q, N_G(\langle \rho \rangle)\rangle$ contains a Sylow 2-subgroup of $C_G(z)$.*

Proof. Set $X = C_G(z)$ and $\bar{X} = X/Q$. Let $\bar{E}$ be an elementary abelian subgroup of order 16 in $\bar{H}$ such that $N_{\bar{H}}(\bar{E})$ involves A_6. Application of Lemma 1 (ii) yields $C_{\bar{H}}(\bar{E}) = \bar{E} \times \langle \bar{\rho} \rangle$. Now $C_{\bar{X}}(\bar{E})$ possesses a normal 3-complement $\bar{K}$. The Frattini argument yields $N_{\bar{X}}(\bar{E}) = \bar{K} N_{N_{\bar{X}}(\bar{E})}(\bar{S})$, where $\bar{S}$ is a Sylow 2-subgroup of $\bar{K}$. Therefore $N_{\bar{N}}(\bar{S})$ contains a Sylow 3-subgroup of $\bar{H}$. Let $W = \langle \tau, \rho \rangle$ be an elementary abelian subgroup of $N_X(S)$ of order 9 with $|C_{\bar{E}}(\bar{\tau})| = 4$. Let x be an element in $W - \langle \rho \rangle$. Then Lemma 1 (ii) yields that 8 divides $|C_{N_{\bar{X}}(\bar{E})}(\bar{x})|$. As Wielandt's fixpoint formula yields $|C_Q(x)| = 32$ and $x \sim \rho$ in G, we get $|C_{\bar{S}}(\bar{x})| = 2^2$. Hence $C_{\bar{S}}(\bar{y}) \leqq \bar{E}$ for all $1 \neq \bar{y} \in \bar{W}$. But then

$$\bar S = \langle C_{\bar S}(\bar y), 1 \neq \bar y \in \bar W \rangle \subseteq \bar E.$$

This shows $\bar S = \bar E$.

Let $\bar T$ be a Sylow 2-subgroup of $\bar H$ containing $\bar E$ and $\bar T_1 \leqq \bar X$, $|\bar T_1 : \bar T| = 2$. Set $\bar T_2 = \bar T \cap \overline{C_H(\rho)}$. Then Lemma 4 yields $\bar y^{\bar T_1} \subseteq \bar T_2$ for all involutions $\bar y \in \bar T_2$. By Lemma 1 (ii) we have $\bar T_2 = \Omega_1(\bar T_2)$. Hence $\bar T_1 \leqq N_{\bar X}(\bar T_2)$. Now Lemma 1 (ii) yields $\bar T_1 \leqq N_{\bar N}(\bar E)$. The Frattini argument yields $N_{\bar X}(\bar E) = C_{\bar X}(\bar E) N_{N_{\bar X}(\bar E)}(\langle \bar\rho \rangle)$. As $\bar E$ is a Sylow 2-subgroup of $C_{\bar X}(\bar E)$ we have that $N_{N_{\bar X}(\bar E)}(\langle \bar\rho \rangle)$ contains a Sylow 2-subgroup of $N_{\bar X}(\bar E)$. But $|\bar T_1| = 2^9$. This contradiction proves the lemma.

LEMMA 6. $G \cong J_4$.

PROOF. Set $H = \langle Q, N_G(\langle \rho \rangle)\rangle$. From Lemma 5 we know that $H/Q = \bar H$ contains a Sylow 2-subgroup of $\overline{C_G(z)}$. From Lemma 4 it follows that no involution in $\bar H - \overline{C_H(\rho)}$ is conjugated in $\overline{C_G(z)}$ to an involution in $\overline{C_H(\rho)}$. From Lemma 1 (i) we know that there are involutions in $\bar H - \overline{C_H(\rho)}$. Now [3, (5.38)] yields that $\overline{C_G(z)}$ possesses a subgroup $\bar K$ of index two. A Sylow 2-subgroup of $\bar K$ is of type M_{22}. As $\bar K$ acts faithfully on $Q/\langle z \rangle$, [1] yields $O'(\bar K / O(\bar K)) \cong M_{22}$. But then a Frattini argument yields $\overline{C_G(z)} = O(\overline{C_G(z)})\overline{N_G(\langle \rho \rangle)}$. Further $\bar\rho \in O(\overline{C_G(z)})$. So $O(\overline{C_G(z)})$ possesses a normal 3-complement $\bar X_1$. The order of $O^+(12,2)$ yields that an element of order 11 in $\overline{N_G(\langle \rho \rangle)}$ centralizes $\bar X_1$. But then $\bar X_1 \leqq \overline{N_G(\langle \rho \rangle)}$. This yields $H = C_G(z)$. Clearly G is simple. Now [2] yields the assertion.

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